Math 4550
Homework 3
Solutions

1

(a) $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}),$
$(\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}) \}$

(b) $(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\underset{\text{in } \mathbb{Z}_2}{\bar{2}}, \underset{\text{in } \mathbb{Z}_3}{\bar{2}}) = (\bar{0},\bar{2})$

(c) $(\bar{1},\bar{1})+(\bar{1},\bar{2}) = (\bar{2},\bar{3}) = (\bar{0},\bar{0})$

So, $(\bar{1},\bar{2})$ is the inverse of $(\bar{1},\bar{1})$.

(d)

$(\bar{0},\bar{2})$

$(\bar{0},\bar{2})+(\bar{0},\bar{2}) = (\bar{0},\bar{4}) = (\bar{0},\bar{1})$

$(\bar{0},\bar{2})+(\bar{0},\bar{2})+(\bar{0},\bar{2}) = (\bar{0},\bar{6}) = (\bar{0},\bar{0})$

Thus, $(\bar{0},\bar{2})$ has order 3 in $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(e)

$\boxed{(\bar{1},\bar{1})}$

$(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\bar{2},\bar{2}) = \boxed{(\bar{0},\bar{2})}$

$$(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\bar{3},\bar{3}) = \boxed{(\bar{1},\bar{0})}$$

in $\mathbb{Z}_2$ in $\mathbb{Z}_3$

$$(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\bar{4},\bar{4}) = \boxed{(\bar{0},\bar{1})}$$

$$(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\bar{5},\bar{5}) = \boxed{(\bar{1},\bar{2})}$$

in $\mathbb{Z}_2$ in $\mathbb{Z}_3$

$$(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1})+(\bar{1},\bar{1}) = (\bar{6},\bar{6}) = \boxed{(\bar{0},\bar{0})}$$

Thus,

$$\langle(\bar{1},\bar{1})\rangle = \{(\bar{1},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{0}),(\bar{0},\bar{1}),(\bar{1},\bar{2}),(\bar{0},\bar{0})\}$$

$$= \mathbb{Z}_2 \times \mathbb{Z}_3.$$

So, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

(2)

(a) $(\bar{2},\bar{8})+(\bar{3},\bar{7}) = (\bar{5},\overline{15}) = (\bar{1},\bar{3})$

in $\mathbb{Z}_4$ in $\mathbb{Z}_{12}$

(b) $(\bar{3},\bar{5})+(\bar{1},\overline{11}) = (\bar{4},\overline{16}) = (\bar{0},\bar{4})$

in $\mathbb{Z}_4$ in $\mathbb{Z}_{12}$

(c) $(\bar{3},\bar{5})+(\bar{1},\bar{7}) = (\bar{4},\overline{12}) = (\bar{0},\bar{0})$

in $\mathbb{Z}_4$ in $\mathbb{Z}_{12}$

Thus, $(\bar{1},\bar{7})$ is the inverse of $(\bar{3},\bar{5})$.

(d) $(\bar{2},\bar{3})$

$$(\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{4},\bar{6}) = (\bar{0},\bar{6})$$

$$(\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{6},\bar{9}) = (\bar{2},\bar{9})$$

$$(\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3})+(\bar{2},\bar{3}) = (\bar{8},\overline{12}) = (\bar{0},\bar{0})$$

Thus, $\langle(\bar{2},\bar{3})\rangle = \{(\bar{2},\bar{3}),(\bar{0},\bar{6}),(\bar{2},\bar{9}),(\bar{0},\bar{0})\}$

and $(\bar{2},\bar{3})$ has order 4.

3. These are done in $\mathbb{Z}_3 \times D_6$ where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and $D_6 = \{1, r, r^2, s, sr, sr^2\}$ and $r^3 = 1$, $s^2 = 1$.

(a) $(\bar{2}, sr)(\bar{2}, sr^2) = (\bar{2} + \bar{2}, srsr^2)$

$$= (\bar{4}, ssr^{-1}r^2) = (\bar{1}, r)$$

(b) $(\bar{1}, r^2)(\bar{0}, sr) = (\bar{1} + \bar{0}, r^2 sr)$

$$= (\bar{1}, sr^{-2}r) = (\bar{1}, sr^{-1})$$

$r^3 = 1$ $$= (\bar{1}, sr^3 r^{-1}) = (\bar{1}, sr^2)$$

(c) $(\bar{1}, r^2)(\bar{2}, r) = (\bar{1} + \bar{2}, r^2 r) = (\bar{3}, r^3) = (\bar{0}, 1)$

Thus, $(\bar{2}, r)$ is the inverse of $(\bar{1}, r^2)$.

(d) $(\bar{1}, r)$

$(\bar{1}, r)(\bar{1}, r) = (\bar{1} + \bar{1}, r^2) = (\bar{2}, r^2)$

$(\bar{1}, r)(\bar{1}, r)(\bar{1}, r) = (\bar{1} + \bar{1} + \bar{1}, rrr) = (\bar{3}, r^3) = (\bar{0}, 1)$

Thus, $(\bar{1}, r)$ has order 3.

④

$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{0},\bar{3}),$
$(\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2}), (\bar{1},\bar{3}) \}$

$H = \{ (\bar{0},\bar{0}), (\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{1},\bar{2}) \}$

Let's use a table to show that $H \trianglelefteq \mathbb{Z}_2 \times \mathbb{Z}_4$

H	$(\bar{0},\bar{0})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{2})$
$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{2})$
$(\bar{0},\bar{2})$	$(\bar{0},\bar{2})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{2})$	$(\bar{1},\bar{0})$
$(\bar{1},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{2})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{2})$
$(\bar{1},\bar{2})$	$(\bar{1},\bar{2})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{2})$	$(\bar{0},\bar{0})$

We have

① $(\bar{0},\bar{0}) \in H$

② H is closed under $+$

③ H is closed under inversion, note that each element is it's own inverse.

By ①, ②, ③ we have that $H \trianglelefteq \mathbb{Z}_2 \times \mathbb{Z}_4$

From the table we see that $(\bar{0},\bar{0})$ has order 1 and $(\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{1},\bar{2})$ each have order 2. Thus, since H has size 4 it does not have a generator and is not cyclic.

(5)

Let $x, y \in G_1 \times G_2$.

Then $x = (a_1, a_2)$ and $y = (b_1, b_2)$

where $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

Then,

$$
\begin{aligned}
xy &= (a_1, a_2)(b_1, b_2) \\
&= (a_1 a_2, b_1 b_2) \\
&= (a_2 a_1, b_2 b_1) \\
&= (a_2, b_2)(a_1, b_1) \\
&= yx
\end{aligned}
$$

since G_1 and G_2 are abelian

Thus, $G_1 \times G_2$ is abelian. ∎

$\boxed{6}$

Suppose that $G \times G$ is cyclic and that $G \times G = \langle (a,b) \rangle$ where $a, b \in G$.

Let's show that $G = \langle a \rangle$ and thus G is cyclic.

Let $x \in G$.

Then $(x,x) \in G \times G$.

So, $(x,x) = (a,b)^n$ for some $n \in \mathbb{Z}$.

Thus, $(x,x) = (a^n, b^n)$.

So, $x = a^n$.

Thus, $G = \langle a \rangle$. $\boxed{//}$