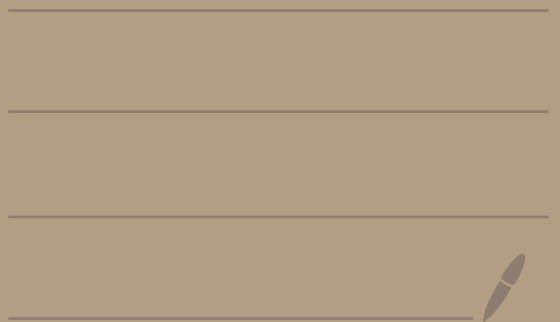


Math 4550

Homework 3

Solutions



(1)

$$(a) \mathbb{Z}_2 \times \mathbb{Z}_3 = \{ (\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{0}, \bar{2}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1}), (\bar{1}, \bar{2}) \}$$

$$(b) (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{2}, \bar{2}) = (\bar{0}, \bar{2})$$

$\uparrow$ in $\mathbb{Z}_2$ $\uparrow$ in $\mathbb{Z}_3$

$$(c) (\bar{1}, \bar{1}) + (\bar{1}, \bar{2}) = (\bar{2}, \bar{3}) = (\bar{0}, \bar{0})$$

So, $(\bar{1}, \bar{2})$ is the inverse of $(\bar{1}, \bar{1})$.

(d)

$$(\bar{0}, \bar{2})$$

$$(\bar{0}, \bar{2}) + (\bar{0}, \bar{2}) = (\bar{0}, \bar{4}) = (\bar{0}, \bar{1})$$

$$(\bar{0}, \bar{2}) + (\bar{0}, \bar{2}) + (\bar{0}, \bar{2}) = (\bar{0}, \bar{6}) = (\bar{0}, \bar{0})$$

Thus, $(\bar{0}, \bar{2})$ has order 3 in $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(e)

$$(\bar{1}, \bar{1})$$

$$(\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{2}, \bar{2}) = (\bar{0}, \bar{2})$$

$$(\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{3}, \bar{3}) = (\bar{1}, \bar{0})$$

$$\begin{array}{c} \uparrow \quad \uparrow \\ \text{in } \mathbb{Z}_2 \quad \text{in } \mathbb{Z}_3 \\ \downarrow \quad \downarrow \end{array}$$

$$(\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{4}, \bar{4}) = (\bar{0}, \bar{1})$$

$$(\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{5}, \bar{5}) = (\bar{1}, \bar{2})$$

$$\begin{array}{c} \uparrow \quad \uparrow \\ \text{in } \mathbb{Z}_2 \quad \text{in } \mathbb{Z}_3 \\ \downarrow \quad \downarrow \end{array}$$

$$(\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) + (\bar{1}, \bar{1}) = (\bar{6}, \bar{6}) = (\bar{0}, \bar{0})$$

Thus,

$$\langle (\bar{1}, \bar{1}) \rangle = \{ (\bar{1}, \bar{1}), (\bar{0}, \bar{2}), (\bar{1}, \bar{0}), (\bar{0}, \bar{1}), (\bar{1}, \bar{2}), (\bar{0}, \bar{0}) \}$$

$$= \mathbb{Z}_2 \times \mathbb{Z}_3.$$

So, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

(2)

$\text{in } \mathbb{Z}_4$ $\text{in } \mathbb{Z}_{12}$

(a) $(\bar{2}, \bar{8}) + (\bar{3}, \bar{7}) = (\bar{5}, \bar{15}) = (\bar{1}, \bar{3})$

(b) $(\bar{3}, \bar{5}) + (\bar{1}, \bar{11}) = (\bar{4}, \bar{16}) = (\bar{0}, \bar{4})$
 $\text{in } \mathbb{Z}_4$ $\text{in } \mathbb{Z}_{12}$

(c) $(\bar{3}, \bar{5}) + (\bar{1}, \bar{7}) = (\bar{4}, \bar{12}) = (\bar{0}, \bar{0})$
 $\text{in } \mathbb{Z}_4$ $\text{in } \mathbb{Z}_{12}$

Thus, $(\bar{1}, \bar{7})$ is the inverse of $(\bar{3}, \bar{5})$.

(d) $(\bar{2}, \bar{3})$

$(\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) = (\bar{4}, \bar{6}) = (\bar{0}, \bar{6})$

$(\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) = (\bar{6}, \bar{9}) = (\bar{2}, \bar{9})$

$(\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) + (\bar{2}, \bar{3}) = (\bar{8}, \bar{12}) = (\bar{0}, \bar{0})$

Thus, $\langle (\bar{2}, \bar{3}) \rangle = \{ (\bar{2}, \bar{3}), (\bar{0}, \bar{6}), (\bar{2}, \bar{9}), (\bar{0}, \bar{0}) \}$

and $(\bar{2}, \bar{3})$ has order 4.

③ These are done in $\mathbb{Z}_3 \times D_6$
 where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ and $D_6 = \{1, r, r^2, s, sr, sr^2\}$
 and $r^3 = 1, s^2 = 1$.

(a) $(\bar{2}, sr)(\bar{2}, sr^2) = (\bar{2} + \bar{2}, sr sr^2)$
 $= (\bar{4}, s sr^{-1} r^2) = (\bar{1}, r)$

(b) $(\bar{1}, r^2)(\bar{0}, sr) = (\bar{1} + \bar{0}, r^2 sr)$
 $= (\bar{1}, sr^{-2} r) = (\bar{1}, sr^{-1})$
 $= (\bar{1}, sr^3 r^{-1}) = (\bar{1}, sr^2)$

$r^3 = 1$

(c) $(\bar{1}, r^2)(\bar{2}, r) = (\bar{1} + \bar{2}, r^2 r) = (\bar{3}, r^3) = (\bar{0}, 1)$
 Thus, $(\bar{2}, r)$ is the inverse of $(\bar{1}, r^2)$.

(d) $(\bar{1}, r)$
 $(\bar{1}, r)(\bar{1}, r) = (\bar{1} + \bar{1}, r^2) = (\bar{2}, r^2)$
 $(\bar{1}, r)(\bar{1}, r)(\bar{1}, r) = (\bar{1} + \bar{1} + \bar{1}, r r r) = (\bar{3}, r^3) = (\bar{0}, 1)$
 Thus, $(\bar{1}, r)$ has order 3.

④

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{0}, \bar{2}), (\bar{0}, \bar{3}), (\bar{1}, \bar{0}), (\bar{1}, \bar{1}), (\bar{1}, \bar{2}), (\bar{1}, \bar{3})\}$$

$$H = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{2}), (\bar{1}, \bar{0}), (\bar{1}, \bar{2})\}$$

Let's use a table to show that $H \trianglelefteq \mathbb{Z}_2 \times \mathbb{Z}_4$

H	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{2})$
$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{2})$
$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$
$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{2})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{2})$
$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{2})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{2})$	$(\bar{0}, \bar{0})$

We have

- ① $(\bar{0}, \bar{0}) \in H$
- ② H is closed under $+$
- ③ H is closed under inversion, note that each element is its own inverse.

By ①, ②, ③ we have that $H \trianglelefteq \mathbb{Z}_2 \times \mathbb{Z}_4$

From the table we see that $(\bar{0}, \bar{0})$ has order 1 and $(\bar{0}, \bar{2}), (\bar{1}, \bar{0}), (\bar{1}, \bar{2})$ each have order 2. Thus, since H has size 4 it does not have a generator and is not cyclic.

⑤

Let $x, y \in G_1 \times G_2$.

Then $x = (a_1, a_2)$ and $y = (b_1, b_2)$
where $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

Then,

$$xy = (a_1, a_2)(b_1, b_2)$$

$$= (a_1 a_2, b_1 b_2)$$

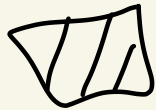
$$= (a_2 a_1, b_2 b_1)$$

Since G_1
and G_2
are abelian

$$= (a_2, b_2)(a_1, b_1)$$

$$= yx$$

Thus, $G_1 \times G_2$ is abelian.



⑥

Suppose that $G \times G$ is cyclic and that
 $G \times G = \langle (a, b) \rangle$ where $a, b \in G$.

Let's show that $G = \langle a \rangle$ and thus
 G is cyclic.

Let $x \in G$.

Then $(x, x) \in G \times G$.

So, $(x, x) = (a, b)^n$ for some $n \in \mathbb{Z}$.

Thus, $(x, x) = (a^n, b^n)$.

So, $x = a^n$.

Thus, $G = \langle a \rangle$.

